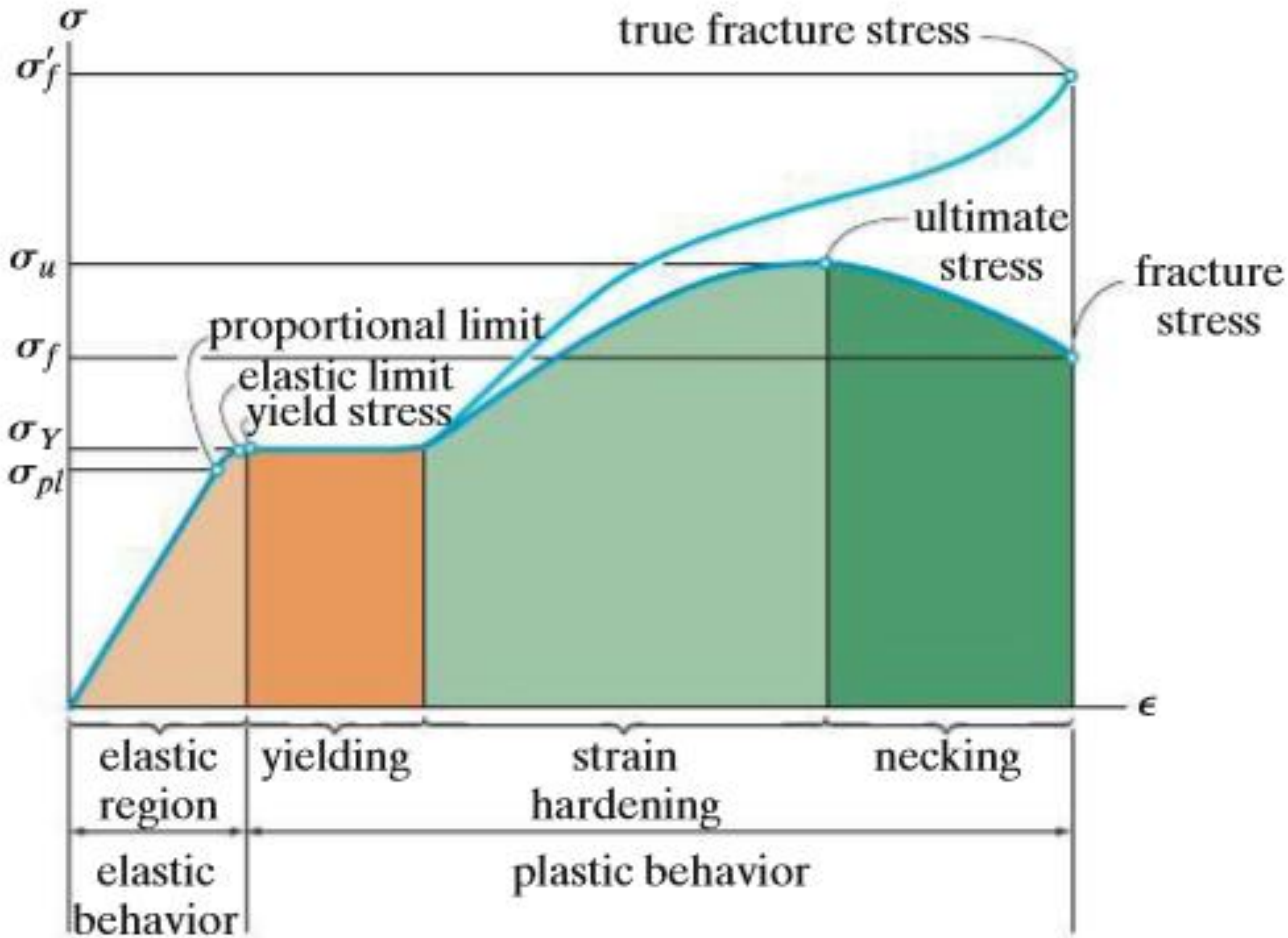
Lecture 5

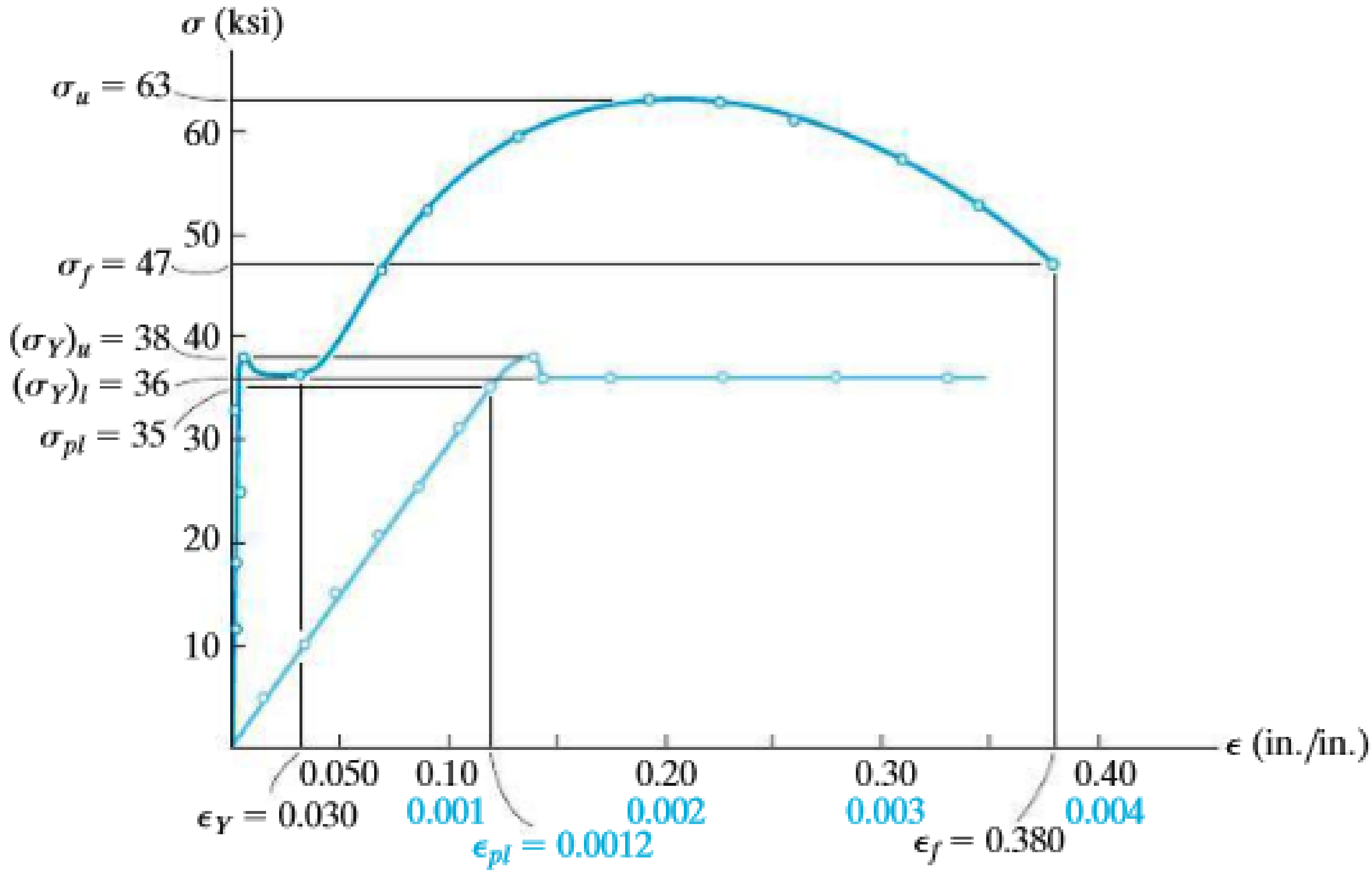


Conventional and true stress-strain diagrams
for ductile material (steel) (not to scale)

True Stress–Strain Diagram. Instead of always using the *original* cross-sectional area and specimen length to calculate the (engineering) stress and strain, we could have used the *actual* cross-sectional area and specimen length at the *instant* the load is measured. The values of stress and strain found from these measurements are called *true stress* and *true strain*, and a plot of their values is called the ***true stress–strain diagram***. When this diagram is plotted it has a form shown by the light-blue curve in Fig. 3–4. Note that the conventional and true σ–ϵ diagrams are practically coincident when the strain is small. The differences between the diagrams begin to appear in the strain-hardening range, where the magnitude of strain becomes more significant. In particular, there is a large divergence within the necking region. Here it can be seen from the conventional σ–ϵ diagram that the specimen *actually* supports a *decreasing load*, since A_0 is constant when calculating engineering stress, $\sigma = P/A_0$. However, from the true σ–ϵ diagram, the actual area A within the necking region is always decreasing until fracture, σ_f', and so the material actually sustains *increasing stress*, since $\sigma = P/A$.

Although the true and conventional stress–strain diagrams are different, most engineering design is done so that the material supports a stress within the elastic range. This is because the deformation of the material is generally not severe and the material will restore itself when the load is removed. The true strain up to the elastic limit will remain small enough so that the error in using the engineering values of σ and ϵ is very small (about 0.1%) compared with their true values. This is one of the primary reasons for using conventional stress–strain diagrams.

The above concepts can be summarized with reference to Fig. 3–6, which shows an actual conventional stress–strain diagram for a mild steel specimen. In order to enhance the details, the elastic region of the curve has been shown in light blue color using an exaggerated strain scale, also shown in light blue. Tracing the behavior, the proportional limit is reached at $\sigma_{pl} = 35$ ksi (241 MPa), where $\epsilon_{pl} = 0.0012$ in./in. This is followed by an upper yield point of $(\sigma_Y)_u = 38$ ksi (262 MPa), then suddenly a lower yield point of $(\sigma_Y)_l = 36$ ksi (248 MPa). The end of yielding occurs at a strain of $\epsilon_Y = 0.030$ in./in., which is 25 times greater than the strain at the proportional limit! Continuing, the specimen undergoes strain hardening until it reaches the ultimate stress of $\sigma_u = 63$ ksi (434 MPa), then it begins to neck down until a fracture occurs, $\sigma_f = 47$ ksi (324 MPa). By comparison, the strain at failure, $\epsilon_f = 0.380$ in./in., is 317 times greater than ϵ_{pl}!



Stress-strain diagram for mild steel

3.3 Stress–Strain Behavior of Ductile and Brittle Materials

Materials can be classified as either being ductile or brittle, depending on their stress–strain characteristics.

Ductile Materials. Any material that can be subjected to large strains before it fractures is called a ***ductile material***. Mild steel, as discussed previously, is a typical example. Engineers often choose ductile materials for design because these materials are capable of absorbing shock or energy, and if they become overloaded, they will usually exhibit large deformation before failing.

One way to specify the ductility of a material is to report its percent elongation or percent reduction in area at the time of fracture. The ***percent elongation*** is the specimen's fracture strain expressed as a percent. Thus, if the specimen's original gauge length is L_0 and its length at fracture is L_f, then

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%) \qquad (3\text{–}3)$$

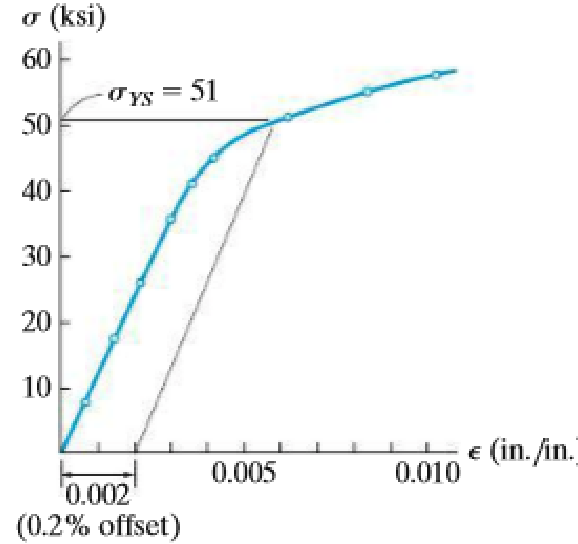
As seen in Fig. 3–6, since $\epsilon_f = 0.380$, this value would be 38% for a mild steel specimen.

The ***percent reduction in area*** is another way to specify ductility. It is defined within the region of necking as follows:

$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%) \qquad (3\text{–}4)$$

Here A_0 is the specimen's original cross-sectional area and A_f is the area of the neck at fracture. Mild steel has a typical value of 60%.

Besides steel, other metals such as brass, molybdenum, and zinc may also exhibit ductile stress–strain characteristics similar to steel, whereby they undergo elastic stress–strain behavior, yielding at constant stress, strain hardening, and finally necking until fracture. In most metals, however, constant yielding will *not occur* beyond the elastic range. One metal for which this is the case is aluminum. Actually, this metal often does not have a well-defined *yield point*, and consequently it is standard practice to define a ***yield strength*** using a graphical procedure called the ***offset method***. Normally for structural design a 0.2% strain (0.002 in./in.) is chosen, and from this point on the ϵ axis, a line parallel to the initial straight-line portion of the stress–strain diagram is drawn. The point where this line intersects the curve defines the yield strength. An example of the construction for determining the yield strength for an aluminum alloy is shown in Fig. 3–7. From the graph, the yield strength is $\sigma_{YS} = 51$ ksi (352 MPa). Apart from metals, 0.2% strain is used as the offset to determine the yield strength of many plastics.

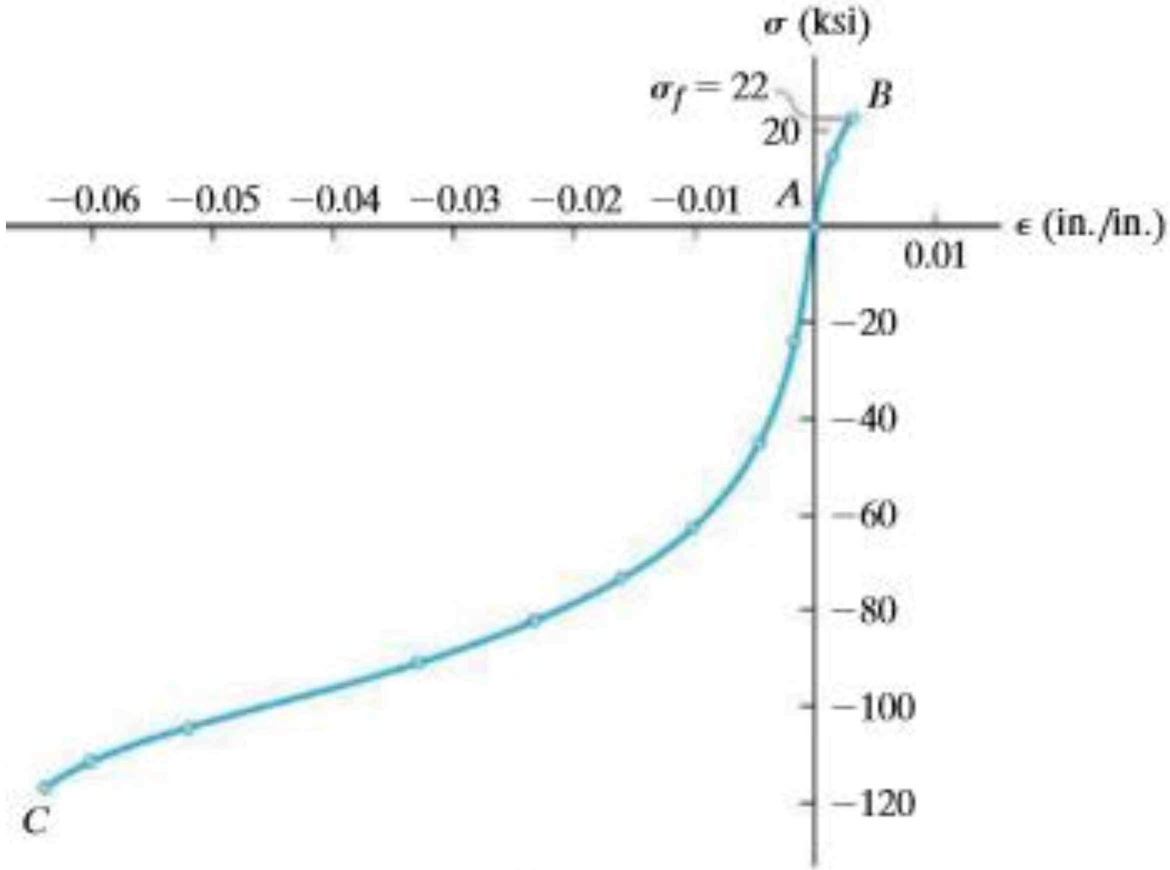


Yield strength for an aluminum alloy

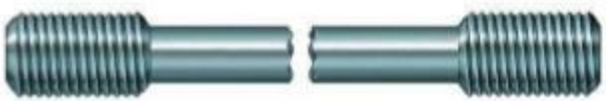
Fig. 3–7

Brittle Materials. Materials that exhibit little or no yielding before failure are referred to as ***brittle materials***. Gray cast iron is an example, having a stress–strain diagram in tension as shown by portion *AB* of the curve in Fig. 3–9. Here fracture at $\sigma_f = 22$ ksi (152 MPa) took place initially at an imperfection or microscopic crack and then spread rapidly across the specimen, causing complete fracture. Since the appearance of initial cracks in a specimen is quite random, brittle materials do not have a well-defined tensile fracture stress. Instead the *average* fracture stress from a set of observed tests is generally reported. A typical failed specimen is shown in Fig. 3–10*a*.

Compared with their behavior in tension, brittle materials, such as gray cast iron, exhibit a much higher resistance to axial compression, as evidenced by portion *AC* of the curve in Fig. 3–9. For this case any cracks or imperfections in the specimen tend to close up, and as the load increases the material will generally bulge or become barrel shaped as the strains become larger, Fig. 3–10*b*.

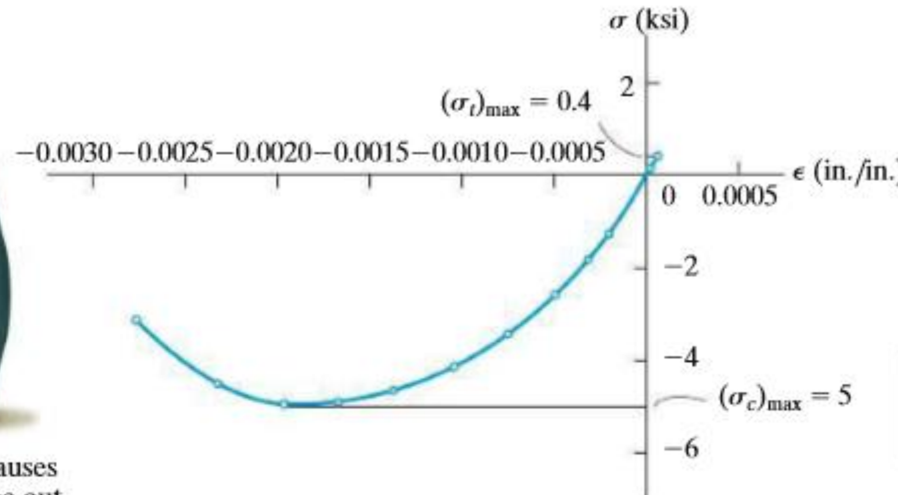


σ–ϵ diagram for gray cast iron



Tension failure of
a brittle material
(a)

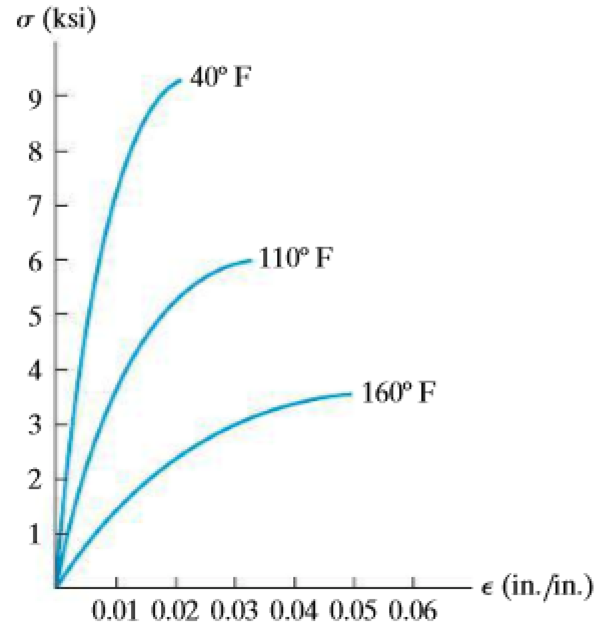
Compression causes
material to bulge out
(b)



$\sigma - \epsilon$ diagram for typical concrete mix increases the material will generally bulge or become barrel shaped as the strains become larger, Fig. 3–10*b*.

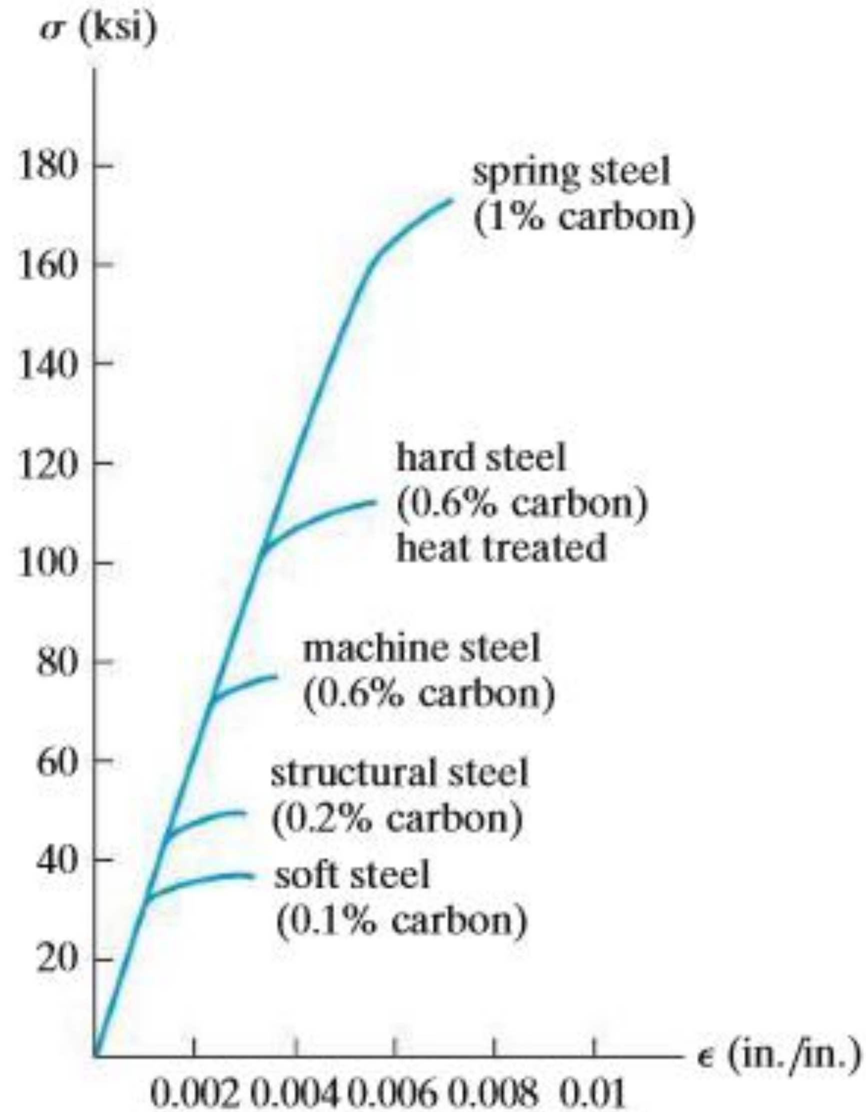
Like gray cast iron, concrete is classified as a brittle material, and it also has a low strength capacity in tension. The characteristics of its stress–strain diagram depend primarily on the mix of concrete (water, sand, gravel, and cement) and the time and temperature of curing. A typical example of a "complete" stress–strain diagram for concrete is given in Fig. 3–11. By inspection, its maximum compressive strength is about 12.5 times greater than its tensile strength, $(\sigma_c)_{\max} = 5$ ksi (34.5 MPa) versus $(\sigma_t)_{\max} = 0.40$ ksi (2.76 MPa). For this reason, concrete is almost always reinforced with steel bars or rods whenever it is designed to support tensile loads.

It can generally be stated that most materials exhibit both ductile and brittle behavior. For example, steel has brittle behavior when it contains a high carbon content, and it is ductile when the carbon content is reduced. Also, at low temperatures materials become harder and more brittle, whereas when the temperature rises they become softer and more ductile. This effect is shown in Fig. 3–12 for a methacrylate plastic.



$\sigma-\epsilon$ diagrams for a methacrylate plastic

Fig. 3–12



σ (ksi)
180
160
140
120
100
80
60
40
20
spring steel
(1% carbon)
hard steel
(0.6% carbon)
heat treated
machine steel
(0.6% carbon)
structural steel
(0.2% carbon)
soft steel
(0.1% carbon)
ϵ (in./in.)
0.002
0.004
0.006
0.008
0.01

3.4 Hooke's Law

As noted in the previous section, the stress–strain diagrams for most engineering materials exhibit a *linear relationship* between stress and strain within the elastic region. Consequently, an increase in stress causes a proportionate increase in strain. This fact was discovered by Robert Hooke in 1676 using springs and is known as *Hooke's law*. It may be expressed mathematically as

$$\sigma = E\epsilon \qquad (3\text{–}5)$$

Here E represents the constant of proportionality, which is called the ***modulus of elasticity*** or ***Young's modulus***, named after Thomas Young, who published an account of it in 1807.

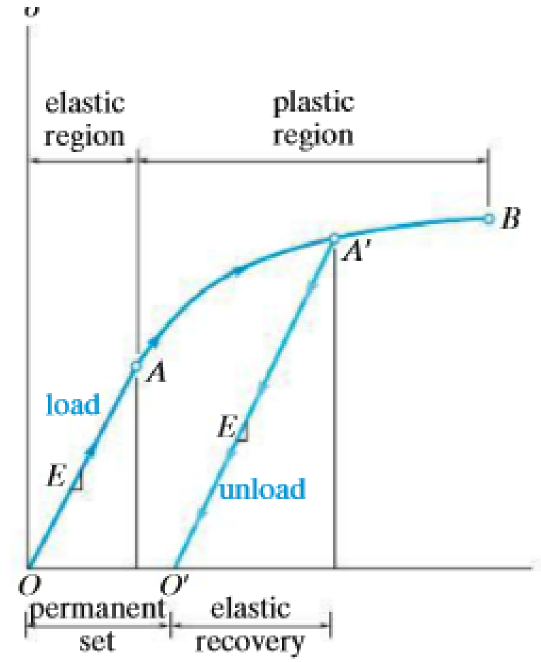
Equation 3–5 actually represents the equation of the *initial straight-lined portion* of the stress–strain diagram up to the proportional limit. Furthermore, the modulus of elasticity represents the *slope* of this line. Since strain is dimensionless, from Eq. 3–5, E will have the same units as stress, such as psi, ksi, or pascals. As an example of its calculation, consider the stress–strain diagram for steel shown in Fig. 3–6. Here $\sigma_{pl} = 35$ ksi and $\epsilon_{pl} = 0.0012$ in./in., so that

$$E = \frac{\sigma_{pl}}{\epsilon_{pl}} = \frac{35 \text{ ksi}}{0.0012 \text{ in./in.}} = 29(10^3) \text{ ksi}$$

As shown in Fig. 3–13, the proportional limit for a particular type of steel alloy depends on its carbon content; however, most grades of steel, from the softest rolled steel to the hardest tool steel, have about the same modulus of

Strain Hardening. If a specimen of ductile material, such as steel, is loaded into the *plastic region* and then unloaded, *elastic strain is recovered* as the material returns to its equilibrium state. The *plastic strain remains*, however, and as a result the material is subjected to a ***permanent set***. For example, a wire when bent (plastically) will spring back a little (elastically) when the load is removed; however, it will not fully return to its original position. This behavior can be illustrated on the stress–strain diagram shown in Fig. 3–14*a*. Here the specimen is first loaded beyond its yield point A to point A'. Since interatomic forces have to be overcome to elongate the specimen *elastically*, then these same forces pull the atoms back together when the load is removed, Fig. 3–14*a*. Consequently, the modulus of elasticity, E, is the same, and therefore the slope of line $O'A'$ is the same as line OA.

If the load is reapplied, the atoms in the material will again be displaced until yielding occurs at or near the stress A', and the stress–strain diagram continues along the same path as before, Fig. 3–14*b*. It should be noted, however, that this new stress–strain diagram, defined by $O'A'B$, now has a *higher* yield point (A'), a consequence of strain-hardening. In other words, the material now has a *greater elastic region*; however, it has *less ductility*, a smaller plastic region, than when it was in its original state.



(a)

3.5 Strain Energy

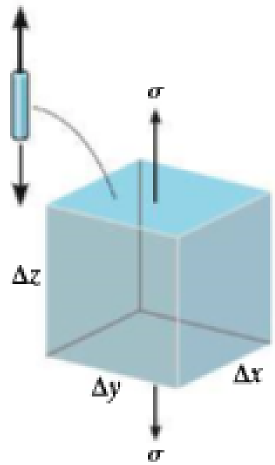


Fig. 3–15

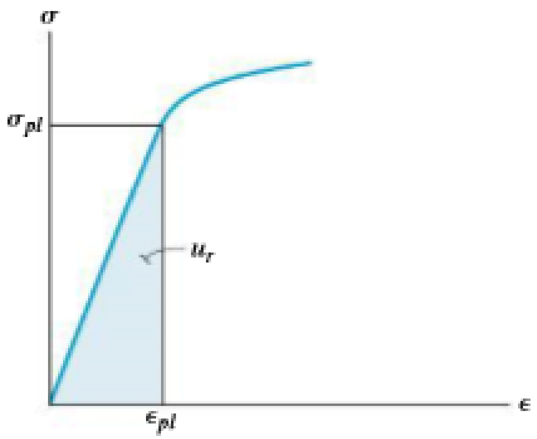
As a material is deformed by an external load, the load will do external work, which in turn will be stored in the material as internal energy. This energy is related to the strains in the material, and so it is referred to as ***strain energy***. To obtain this strain energy let us consider a volume element of material from a tension test specimen Fig. 3–15. It is subjected to the uniaxial stress σ. This stress develops a force $\Delta F = \sigma\, \Delta A = \sigma(\Delta x\, \Delta y)$ on the top and bottom faces of the element *after* the element of length Δz undergoes a vertical displacement $\epsilon\, \Delta z$. By definition, *work* of ΔF is determined by the product of a force and the displacement in the direction of the force. Since the force is increased uniformly from zero to its final magnitude ΔF when the displacement $\epsilon\, \Delta z$ occurs, the work done on the element by the force is then equal to the *average* force magnitude $(\Delta F/2)$ times the displacement $\epsilon\, \Delta z$. The conservation of energy requires this "external work" on the element to be equivalent to the "internal work" or strain energy stored in the element—assuming that no energy is lost in the form of heat. Consequently, the strain energy ΔU is $\Delta U = (\frac{1}{2}\Delta F)\, \epsilon\, \Delta z = (\frac{1}{2}\, \sigma\, \Delta x\, \Delta y)\, \epsilon\, \Delta z$. Since the volume of the element is $\Delta V = \Delta x\, \Delta y\, \Delta z$, then $\Delta U = \frac{1}{2}\, \sigma\epsilon\, \Delta V$.

For applications, it is often convenient to specify the strain energy per unit volume of material. This is called the ***strain-energy density***, and it can be expressed as

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon \tag{3–6}$$

Finally, if the material behavior is *linear elastic*, then Hooke's law applies, $\sigma = E\epsilon$, and therefore we can express the *elastic strain-energy density* in terms of the uniaxial stress σ as

$$u = \frac{1}{2}\frac{\sigma^2}{E} \tag{3–7}$$



Modulus of resilience u_r

(a)

Fig. 3–16

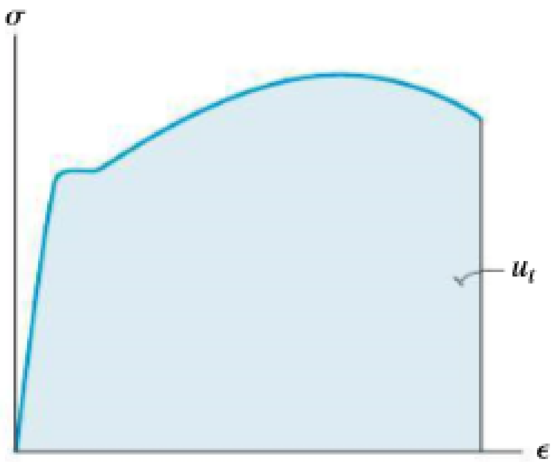
Modulus of Resilience.

In particular, when the stress σ reaches the proportional limit, the strain-energy density, as calculated by Eq. 3–6 or 3–7, is referred to as the ***modulus of resilience***, i.e.,

$$u_r = \frac{1}{2}\sigma_{pl}\,\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E} \qquad (3\text{–}8)$$

From the elastic region of the stress–strain diagram, Fig. 3–16*a*, notice that u_r is equivalent to the shaded *triangular area* under the diagram. Physically the modulus of resilience represents the largest amount of internal strain energy per unit volume the material can absorb without causing any permanent damage to the material. Certainly this becomes important when designing bumpers or shock absorbers.

Modulus of Toughness. Another important property of a material is the ***modulus of toughness***, u_t. This quantity represents the *entire area* under the stress–strain diagram, Fig. 3–16*b*, and therefore it indicates the maximum amount of strain-energy the material can absorb just before it fractures. This property becomes important when designing members that may be accidentally overloaded. Note that alloying metals can also change their resilience and toughness. For example, by changing the percentage of carbon in steel, the resulting stress–strain diagrams in Fig. 3–17 show how the degrees of resilience (Fig. 3–16*a*) and toughness (Fig. 3–16*b*) can be changed.



Modulus of toughness u_t

A tension test for a steel alloy results in the stress–strain diagram shown in Fig. 3–18. Calculate the modulus of elasticity and the yield strength based on a 0.2% offset. Identify on the graph the ultimate stress and the fracture stress.

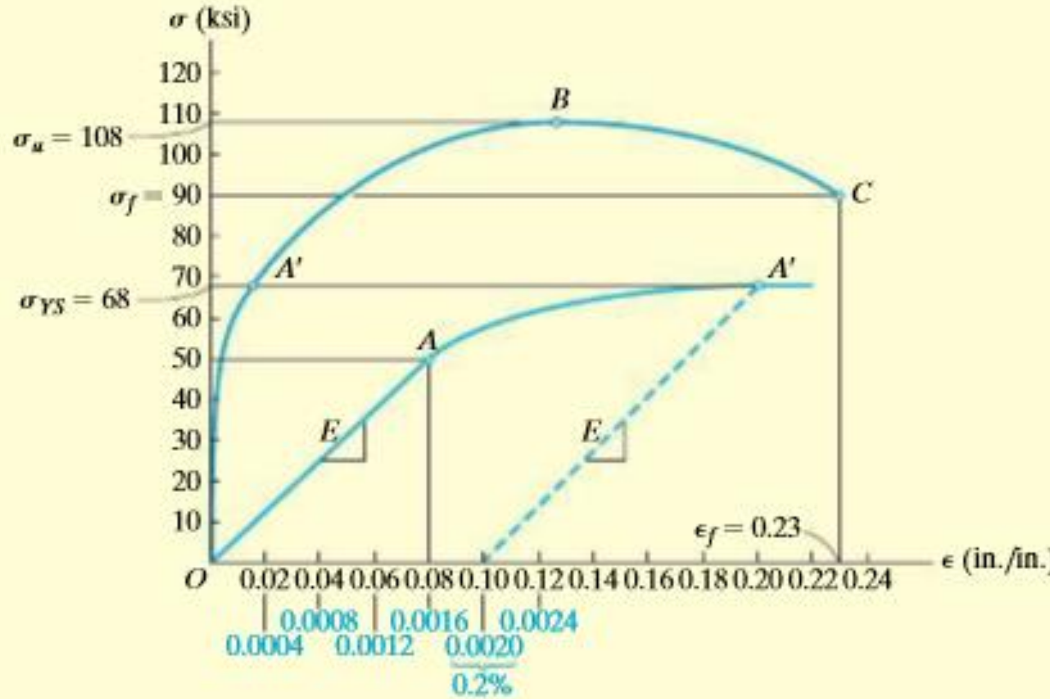


Fig. 3–18

Fig. 3–18

SOLUTION

Modulus of Elasticity. We must calculate the *slope* of the initial straight-line portion of the graph. Using the magnified curve and scale shown in blue, this line extends from point O to an estimated point A, which has coordinates of approximately (0.0016 in./in., 50 ksi). Therefore,

$$E = \frac{50 \text{ ksi}}{0.0016 \text{ in./in.}} = 31.2(10^3) \text{ ksi} \qquad \textit{Ans.}$$

Note that the equation of line OA is thus $\sigma = 31.2(10^3)\epsilon$.

Yield Strength. For a 0.2% offset, we begin at a strain of 0.2% or 0.0020 in./in. and graphically extend a (dashed) line parallel to OA until it intersects the σ–ϵ curve at A'. The yield strength is approximately

$$\sigma_{YS} = 68 \text{ ksi} \qquad \textit{Ans.}$$

Ultimate Stress. This is defined by the peak of the σ–ϵ graph, point B in Fig. 3–18.

$$\sigma_u = 108 \text{ ksi} \qquad \textit{Ans.}$$

Fracture Stress. When the specimen is strained to its maximum of $\epsilon_f = 0.23$ in./in., it fractures at point C. Thus,

$$\sigma_f = 90 \text{ ksi} \qquad \textit{Ans.}$$

The stress–strain diagram for an aluminum alloy that is used for making aircraft parts is shown in Fig. 3–19. If a specimen of this material is stressed to 600 MPa, determine the permanent strain that remains in the specimen when the load is released. Also, find the modulus of resilience both before and after the load application.

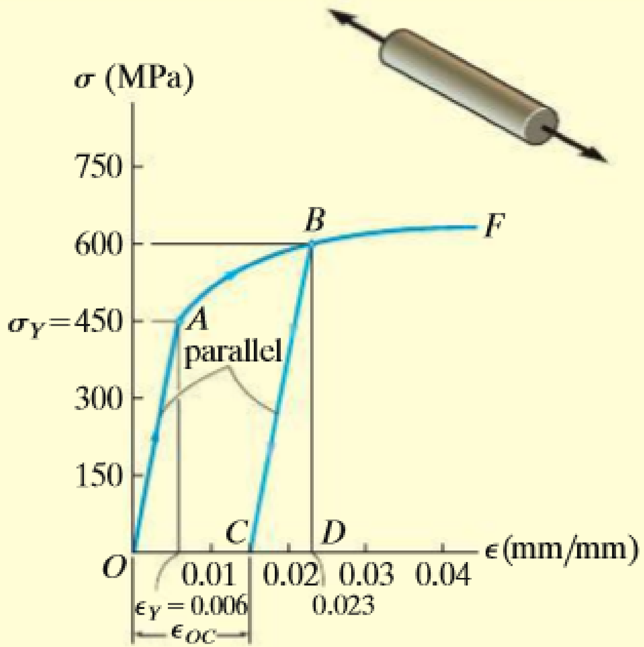


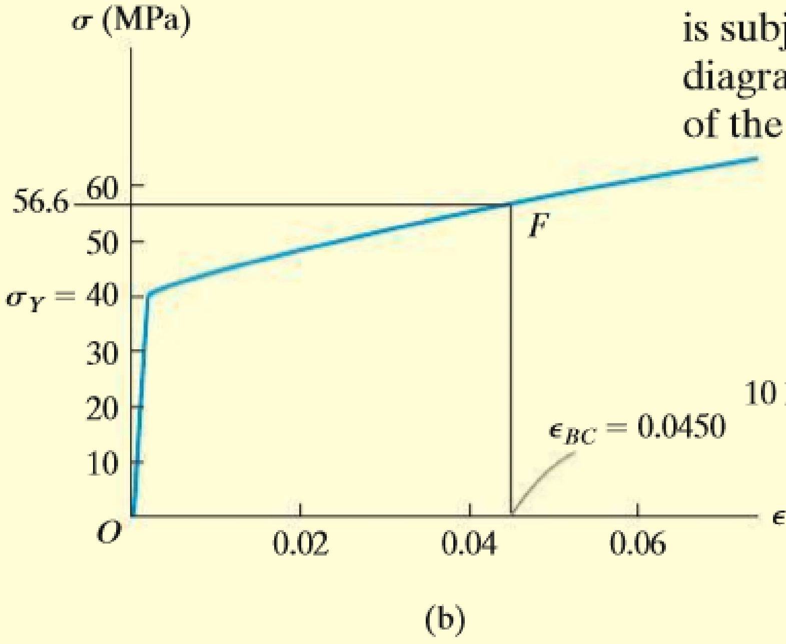
Fig. 3–19

SOLUTION

Permanent Strain. When the specimen is subjected to the load, it strain-hardens until point B is reached on the σ–ϵ diagram. The strain at this point is approximately 0.023 mm/mm. When the load is released, the material behaves by following the straight line BC, which is parallel to line OA. Since both lines have the same slope, the strain at point C can be determined analytically. The slope of line OA is the modulus of elasticity, i.e.,

$$E = \frac{450 \text{ MPa}}{0.006 \text{ mm/mm}} = 75.0 \text{ GPa}$$

An aluminum rod, shown in Fig. 3–20*a*, has a circular cross section and is subjected to an axial load of 10 kN. If a portion of the stress–strain diagram is shown in Fig. 3–20*b*, determine the approximate elongation of the rod when the load is applied. Take $E_{al} = 70$ GPa.



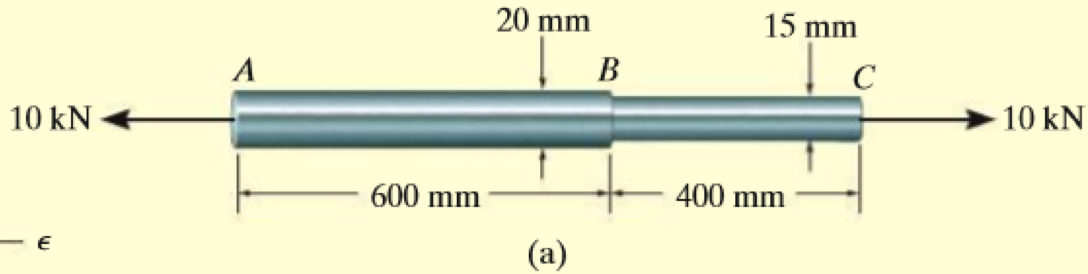
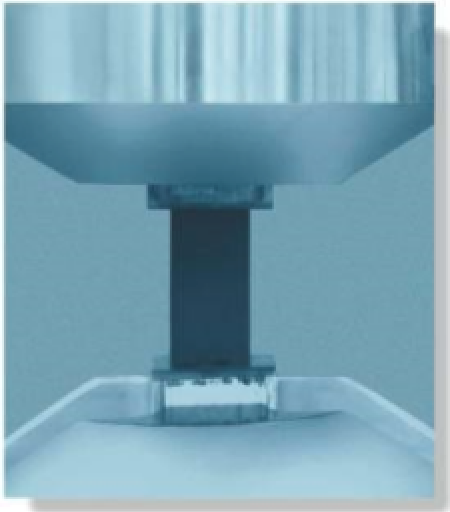
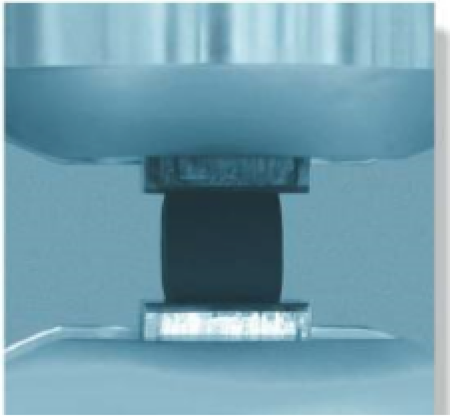


Fig. 3–20

3.6 Poisson's Ratio





When a deformable body is subjected to an axial tensile force, not only does it elongate but it also contracts laterally. For example, if a rubber band is stretched, it can be noted that both the thickness and width of the band are decreased. Likewise, a compressive force acting on a body causes it to contract in the direction of the force and yet its sides expand laterally.

Consider a bar having an original radius r and length L and subjected to the tensile force P in Fig. 3–21. This force elongates the bar by an amount δ, and its radius contracts by an amount δ'. Strains in the longitudinal or axial direction and in the lateral or radial direction are, respectively,

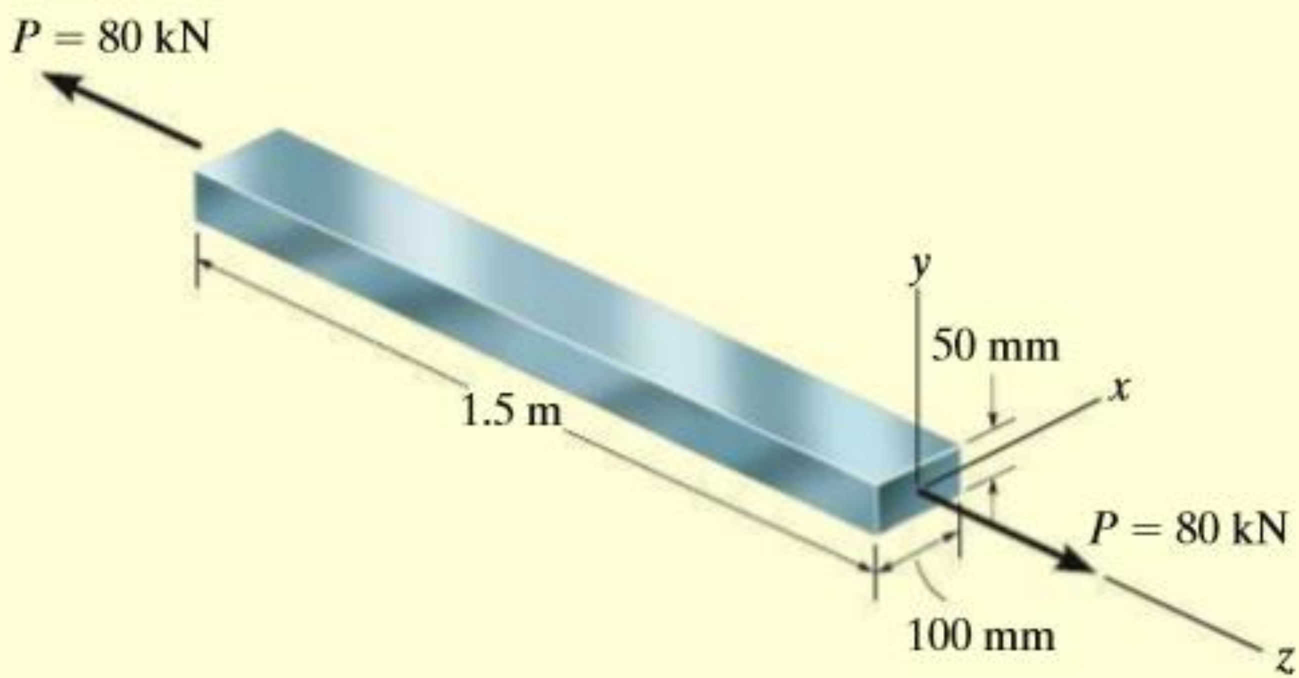
$$\epsilon_{\text{long}} = \frac{\delta}{L} \quad \text{and} \quad \epsilon_{\text{lat}} = \frac{\delta'}{r}$$

In the early 1800s, the French scientist S. D. Poisson realized that within the *elastic range* the *ratio* of these strains is a *constant*, since the deformations δ and δ' are proportional. This constant is referred to as ***Poisson's ratio***, ν (nu), and it has a numerical value that is unique for a particular material that is both *homogeneous and isotropic*. Stated mathematically it is

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}} \tag{3–9}$$

The negative sign is included here since *longitudinal elongation* (positive strain) causes *lateral contraction* (negative strain), and vice versa. Notice that these strains are caused only by the axial or longitudinal force P; i.e., no force or stress acts in a lateral direction in order to strain the material in this direction.

A bar made of A-36 steel has the dimensions shown in Fig. 3–22. If an axial force of $P = 80$ kN is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.



3.7 The Shear Stress–Strain Diagram

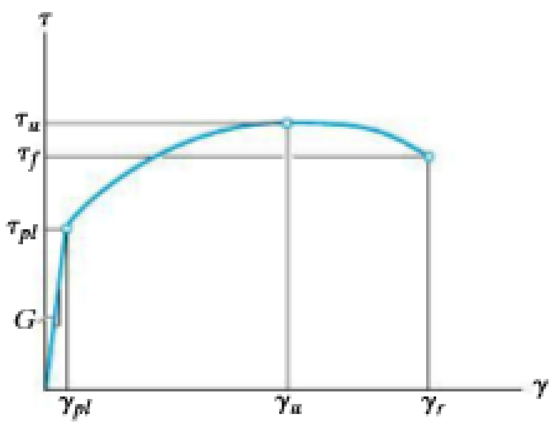


Fig. 3–24

For most engineering materials, like the one just described, the elastic behavior is *linear*, and so Hooke's law for shear can be written as

$$\tau = G\gamma \tag{3–10}$$

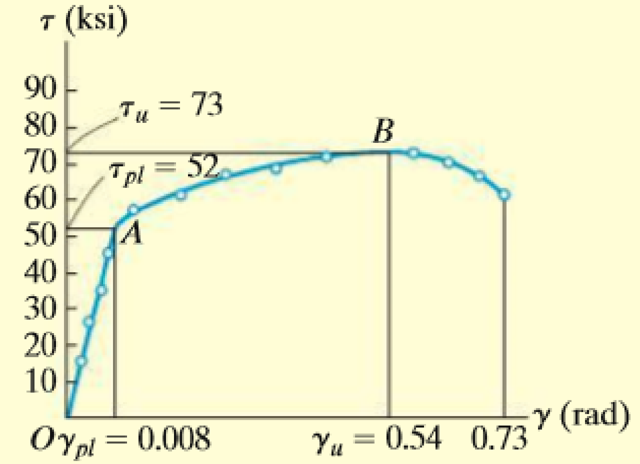
Here G is called the ***shear modulus of elasticity*** or the ***modulus of rigidity***. Its value represents the slope of the line on the τ–γ diagram, that is, $G = \tau_{pl}/\gamma_{pl}$. Typical values for common engineering materials are listed on the inside back cover. Notice that the units of measurement for G will be the *same* as those for τ (Pa or psi), since γ is measured in radians, a dimensionless quantity.

It will be shown in Sec. 10.6 that the three material constants, E, ν, and G are actually *related* by the equation

$$G = \frac{E}{2(1 + \nu)} \tag{3–11}$$

Provided E and G are known, the value of ν can then be determined from this equation rather than through experimental measurement. For example, in the case of A-36 steel, $E_{st} = 29(10^3)$ ksi and $G_{st} = 11(10^3)$ ksi, so that, from Eq. 3–11, $\nu_{st} = 0.32$.

A specimen of titanium alloy is tested in torsion and the shear stress–strain diagram is shown in Fig. 3–25*a*. Determine the shear modulus G, the proportional limit, and the ultimate shear stress. Also, determine the maximum distance d that the top of a block of this material, shown in Fig. 3–25*b*, could be displaced horizontally if the material behaves elastically when acted upon by a shear force **V**. What is the magnitude of **V** necessary to cause this displacement?



An aluminum specimen shown in Fig. 3–26 has a diameter of $d_0 = 25$ mm and a gauge length of $L_0 = 250$ mm. If a force of 165 kN elongates the gauge length 1.20 mm, determine the modulus of elasticity. Also, determine by how much the force causes the diameter of the specimen to contract. Take $G_{al} = 26$ GPa and $\sigma_Y = 440$ MPa.